13.1

a) $$\int_0^1 \frac{x^3}{\sqrt{x^2+1}}\,dx = \int_1^2 \frac{y-1}{\sqrt{y}}\cdot\frac{1}{2}\,dy \qquad \begin{pmatrix} y = x^2+1 \\ dy = 2x\,dx \end{pmatrix}$$

$$= \frac{1}{2}\left(\int_1^2 \sqrt{y}\,dy - \int_1^2 \frac{1}{\sqrt{y}}\,dy\right)$$

$$= \frac{1}{2}\left(\left[\frac{2}{3}y^{\frac{3}{2}}\right]_{y=1}^{y=2} - \left[2\sqrt{y}\right]_{y=1}^{y=2}\right)$$

$$= \frac{\cancel{1}}{\cancel{2}}\left(\frac{\cancel{2}^{1}}{3}\sqrt{8} - \frac{\cancel{2}^{1}}{3} - \cancel{2}\sqrt{2} + \cancel{2}^{1}\right)$$

$$= \frac{2}{3}\sqrt{2} - \frac{1}{3} - \sqrt{2} + 1 = \frac{2}{3} - \frac{1}{3}\sqrt{2}$$

$$= \underline{\underline{\frac{1}{3}\left(2-\sqrt{2}\right)}}$$

b) $$\int_0^1 \frac{e^x}{\sqrt{1+e^x}}\,dx = \int_1^e \frac{1}{\sqrt{1+y}}\,dy \qquad \begin{pmatrix} y = e^x \\ dy = e^x\,dx \end{pmatrix}$$

$$= \left[2\sqrt{1+y}\right]_{y=1}^{y=e} = \underline{\underline{2\left(\sqrt{1+e} - \sqrt{2}\right)}}$$

c) $$\int_0^1 \frac{y^2}{\sqrt{1-y^2}}\,dy = \int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{\sqrt{1-\sin^2 x}}\cos x\,dx \qquad \begin{pmatrix} y = \sin x \\ dy = \cos x\,dx \end{pmatrix}$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{\cos x}\cos x\,dx = \int_0^{\frac{\pi}{2}} \sin^2 x\,dx$$

$$= \underbrace{-\left[\sin x\cos x\right]_{x=0}^{x=\frac{\pi}{2}}}_{=0} + \int_0^{\frac{\pi}{2}} \cos^2 x\,dx = \int_0^{\frac{\pi}{2}} \left(1-\sin^2 x\right)dx$$

$$= \int_0^{\frac{\pi}{2}} dx - \int_0^{\frac{\pi}{2}} \sin^2 x\,dx$$

$$\Rightarrow \int_0^1 \frac{y^2}{\sqrt{1-y^2}}\,dy = \int_0^{\frac{\pi}{2}} \sin^2 x\,dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} dx = \underline{\underline{\frac{\pi}{4}}}$$

13.1

$$\begin{pmatrix} y = \sin^2 x \\ dy = 2\sin x\cos x\,dx \end{pmatrix}$$

d) $$\int_0^1 \frac{\sqrt{1-y}}{y-\sqrt{y}}\,dy = \int_0^{\frac{\pi}{2}} \frac{\sqrt{1-\sin^2 x}}{\sin^2 x - \cancel{\sin x}}\, 2\cancel{\sin x}\cos x\,dx$$

$$= 2\int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\sin x - 1}\,dx = -2\int_0^{\frac{\pi}{2}} \frac{1-\sin^2 x}{1-\sin x}\,dx$$

$$= -2\int_0^{\frac{\pi}{2}} (1+\sin x)\,dx = -2\left(\left[x\right]_{x=0}^{x=\frac{\pi}{2}} + \left[-\cos x\right]_{x=0}^{x=\frac{\pi}{2}}\right)$$

$$= -2\left(\frac{\pi}{2} + 1\right) = \underline{\underline{-\pi - 2}}$$

e) $$\int_0^1 \frac{3x^2+2x+1}{x^3+x^2+x+1}\,dx$$

Zuerst benötigen wir eine Partialbruchzerlegung.
Dafür faktorisieren wir x^3+x^2+x+1

Wir bemerken, dass $x=-1$ eine Nullstelle ist und erhalten

$$x^3+x^2+x+1 = (x+1)(x^2+1)$$

Das 2. Polynom (x^2+1) ist irreduzibel.

Wir machen den Ansatz

$$\frac{3x^2+2x+1}{x^3+x^2+x+1} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$$

$$= \frac{Ax^2+A+Bx^2+Bx+Cx+C}{(x+1)(x^2+1)}$$

$$= \frac{(A+B)x^2+(B+C)x+(A+C)}{x^3+x^2+x+1}$$

Daraus ergibt sich das lineare Gleichungssystem

$$\left.\begin{array}{l} A+B=3 \\ B+C=2 \\ A+C=1 \end{array}\right\} \Rightarrow \left.\begin{array}{l} \\ B-A=1 \end{array}\right\} \Rightarrow 2B=4 \Rightarrow B=2$$

Mit $B=2$ folgt nun $A=1$ und $C=0$

$$\Rightarrow \frac{3x^2+2x+1}{x^3+x^2+x+1} = \frac{1}{x+1} + \frac{2x}{x^2+1}$$

$$\Rightarrow \int_0^1 \frac{3x^2+2x+1}{x^3+x^2+x+1}\,dx = \int_0^1 \frac{1}{x+1}\,dx + \int_0^1 \frac{2x}{x^2+1}\,dx$$

$$= \Big[\log(x+1)\Big]_{x=0}^{x=1} + \int_1^2 \frac{1}{y}\,dy \qquad \left(\begin{array}{l} y=x^2+1 \\ dy = 2x\,dx \end{array}\right)$$

$$= \log(2) - \log(1) + \Big[\log(y)\Big]_{y=1}^{y=2}$$

$$= \log(2) + \log(2) - \log(1)$$

$$= 2\log(2) = \underline{\underline{\log(4)}}$$

f) $$\int_0^{\frac{\pi}{4}} \tan = \int_0^{\frac{\pi}{4}} \frac{\sin(x)}{\cos(x)}\,dx = \int_1^{\frac{\sqrt{2}}{2}} \frac{-1}{y}\,dy \qquad \left(\begin{array}{l} y=\cos x \\ dy=-\sin x\,dx \end{array}\right)$$

$$= -\Big[\log\Big]_1^{\frac{\sqrt{2}}{2}} = -\log\left(\frac{\sqrt{2}}{2}\right) = \underline{\underline{\log(\sqrt{2})}}$$

g) $$\int \frac{\exp}{\sqrt{1+\exp}} = \int \frac{1}{\sqrt{1+y}}\,dy \qquad \left(\begin{array}{l} y=e^x \\ dy = e^x\,dx \end{array}\right)$$

$$= 2\sqrt{1+y} + C^{\mathbb{R}}_{loc} = \underline{\underline{\left(x \mapsto 2\sqrt{1+e^x}\right) + C^{\mathbb{R}}_{loc}}}$$

13.1

b) $\int \tan = \int \frac{\sin x}{\cos x}\, dx = \int -\frac{1}{y}\, dy \quad \begin{pmatrix} y = \cos x \\ dy = -\sin x\, dx \end{pmatrix}$

$= -\log|y| + C_{loc} = \underline{\underline{\left(x \mapsto -\log|\cos x|\right) + C_{loc}}}$

13.2

a) $f(z) = \frac{z+1}{z^2 - 3z + 2}$

Ansatz: $f(z) = \frac{a}{z-1} + \frac{b}{z-2}$

$\Rightarrow \frac{a(z-2) + b(z-1)}{z^2 - 3z + 2} = \frac{z+1}{z^2 - 3z + 2}$

$\Rightarrow \begin{array}{l} a + b = 1 \\ -2a - b = 1 \end{array} \Rightarrow \begin{array}{l} -a = 2 \Rightarrow a = -2 \\ b = 3 \end{array}$

$\Rightarrow f(z) = \underline{\underline{-\frac{2}{z-1} + \frac{3}{z-2}}}$

b) Überprüfung:

$= \frac{-2(z-2) + 3(z-1)}{(z-1)(z-2)}$

$= \frac{-2z + 4 + 3z - 3}{z^2 - 3z + 2} = \frac{z+1}{z^2 - 3z + 2}$

c) Gemäss Satz aus der Vorlesung gibt es für jede komplexe rationale Funktion eine eindeutige Partialbruchzerlegung.

13.2

d) $$f(z) = \frac{z^2 - z + 1}{z^3 - 2z^2 + z}$$

$$z^3 - 2z^2 + z = z(z^2 - 2z + 1)$$
$$= z(z-1)^2$$

Ansatz:

$$f(z) = \frac{a}{z} + \frac{b}{z-1} + \frac{c}{(z-1)^2}$$

$$= \frac{a(z-1)^2 + b(z-1)z + cz}{z(z-1)^2}$$

$$= \frac{(a+b)z^2 + (-2a-b+c)z + a}{z^3 - 2z^2 + z}$$

$$\Rightarrow \quad \begin{aligned} a + b &= 1 \\ -2a - b + c &= -1 \\ a &= 1 \end{aligned} \quad \Rightarrow \quad \begin{aligned} a &= 1 \\ b &= 0 \\ c &= 1 \end{aligned}$$

$$\Rightarrow \quad f(z) = \frac{1}{z} + \frac{1}{(z-1)^2}$$

Überprüfung:

$$= \frac{(z-1)^2 + z}{z(z-1)^2} = \frac{z^2 - 2z + 1 + z}{z^3 - 2z^2 + z}$$

$$= \frac{z^2 - z + 1}{z^3 - 2z + z}$$

13.2

e) $$f(z) = \frac{z^3}{z+1}$$

$$z^3 = (z+1)(z^2 - z + 1) - 1$$

$$\Rightarrow \underline{\underline{f(z) = z^2 - z + 1 - \frac{1}{z+1}}}$$

Überprüfung:

$$= \frac{(z^2 - z + 1)(z+1) - 1}{z+1}$$

$$= \frac{z^3 - \cancel{z^2} + \cancel{z} + \cancel{z^2} - \cancel{z} + \cancel{1} - \cancel{1}}{z+1}$$

$$= \frac{z^3}{z+1}$$

f) $$f(z) = \frac{z^3 - 2z^2 + 3}{z^2 - 3z + 2}$$

$$z^3 - 2z^2 + 3 = (z^2 - 3z + 2)(z+1) + z + 1$$

$$\Rightarrow f(z) = z + 1 + \frac{z+1}{z^2 - 3z + 2}$$

$$= z + 1 - \frac{2}{z-1} + \frac{3}{z-2} \quad \text{(Aufgabe (a))}$$

Überprüfung:

$$= \frac{(z+1)(z-1)(z-2) - 2(z-2) + 3(z-1)}{(z-1)(z-2)}$$

$$= \frac{z^3 - 2z^2 - \cancel{z^2} + \cancel{z^2} - \cancel{z} - \cancel{2z} + \cancel{2z} + 2 - \cancel{2z} + 4 + \cancel{3z} - 3}{z^2 - 3z + 2}$$

$$= \frac{z^3 - 2z^2 + 3}{z^2 - 3z + 2}$$

g)

$$f_1(z) = \frac{z+1}{z^2-3z+2} = \frac{z+1}{(z-1)(z-2)}$$

$$= \sum_{i=1}^{2} \sum_{k=1}^{1} \frac{a_{ik}}{(z-z_i)^k} = \frac{a_{11}}{z-1} + \frac{a_{21}}{z-2}$$

$$a_{11} = \lim_{\substack{z \to 1 \\ z \neq 1}} \left(\frac{z+1}{(z-1)(z-2)} - \underbrace{\sum_{j=2}^{1} \frac{a_{1j}}{(z-z_j)^j}}_{=0} \right)(z-1)$$

$$= \lim_{\substack{z \to 1 \\ z \neq 1}} \frac{z+1}{z-2}$$

$$= -2$$

$$a_{21} = \lim_{\substack{z \to 2 \\ z \neq 2}} \left(\frac{z+1}{(z-1)(z-2)} - \underbrace{\sum_{j=2}^{1} \frac{a_{2j}}{(z-2)^j}}_{=0} \right)(z-2)$$

$$= \lim_{\substack{z \to 2 \\ z \neq 2}} \frac{z+1}{z-1} = 3$$

$$\Rightarrow \underline{\underline{f_1(z) = \frac{-2}{z-1} + \frac{3}{z-2}}}$$

g) $f_2(z) = \dfrac{z^2 - z + 1}{z^3 - 2z^2 + z}$

$$= \sum_{i=1}^{2} \sum_{k=1}^{m_i} \frac{a_{ik}}{(z - z_i)^k}$$

$$= \frac{a_{11}}{z} + \frac{a_{21}}{z-1} + \frac{a_{22}}{(z-1)^2}$$

$$a_{11} = \lim_{\substack{z \to 0 \\ z \neq 0}} \frac{z^2 - z + 1}{\cancel{z}(z-1)^2} \cdot \cancel{z} = 1$$

$$a_{22} = \lim_{\substack{z \to 1 \\ z \neq 1}} \frac{z^2 - z + 1}{z\cancel{(z-1)^2}} \cdot \cancel{(z-1)^2} = 1$$

$$a_{21} = \lim_{\substack{z \to 1 \\ z \neq 1}} \left(\frac{z^2 - z + 1}{z(z-1)^{\cancel{2}}} - \frac{a_{22}}{(z-1)^{\cancel{2}}} \right) \cancel{(z-1)}$$

$$= \lim_{\substack{z \to 1 \\ z \neq 1}} \left(\frac{z^2 - z + 1 - z}{z(z-1)} \right)$$

$$= \lim_{\substack{z \to 1 \\ z \neq 1}} \left(\frac{z-1}{z} \right) = 0$$

$$\Rightarrow f_2(z) = \frac{1}{z} + \frac{1}{(z-1)^2}$$

g) $f_3(z) = \frac{z}{(z^2-1)^2} = \frac{z}{(z-1)^2(z+1)^2}$

$$f_3(z) = \sum_{i=1}^{2} \sum_{k=1}^{2} \frac{a_{ik}}{(z-z_i)^k}$$

$$= \frac{a_{11}}{z+1} + \frac{a_{12}}{(z+1)^2} + \frac{a_{21}}{z-1} + \frac{a_{22}}{(z-1)^2}$$

$$a_{12} = \lim_{\substack{z\to -1 \\ z\neq -1}} \left(\frac{z}{(z-1)^2 \cancel{(z+1)^2}} \right) \cancel{(z+1)^2} = -\frac{1}{4}$$

$$a_{11} = \lim_{\substack{z\to -1 \\ z\neq -1}} \left(\frac{z}{(z-1)^2 (z+1)^{\cancel{2}}} - \frac{a_{21}}{(z+1)^{\cancel{2}}} \right) \cancel{(z+1)}$$

$$= \lim_{\substack{z\to -1 \\ z\neq -1}} \left(\frac{z - (z-1)^2 \cdot \left(-\frac{1}{4}\right)}{(z-1)^2 (z+1)} \right)$$

$$= \lim_{\substack{z\to -1 \\ z\neq -1}} \left(\frac{z + \frac{1}{4} z^2 - \frac{1}{2} z + \frac{1}{4}}{(z-1)^2 (z+1)} \right)$$

$$= \lim_{\substack{z\to -1 \\ z\neq -1}} \left(\frac{\frac{1}{4} (z+1)^{\cancel{2}}}{(z-1)^2 \cancel{(z+1)}} \right) = 0$$

$$f_3(z) = -f_3(-z) = \frac{a_{11}}{z-1} - \frac{a_{12}}{(z-1)^2} + \frac{a_{21}}{z+1} - \frac{a_{22}}{(z+1)^2}$$

$$\Rightarrow a_{21} = a_{11} = 0, \quad a_{22} = -a_{12} = \frac{1}{4}$$

$$\Rightarrow \underline{\underline{f_3(z) = \frac{1}{4} \frac{1}{(z-1)^2} - \frac{1}{4} \frac{1}{(z+1)^2}}}$$

Überprüfung:

$$= \frac{1}{4} \left(\frac{\cancel{z^2} + 2z \cancel{+1} - \cancel{z^2} + 2z \cancel{-1}}{(z-1)^2 (z+1)^2} \right) = \frac{z}{(z^2-1)^2}$$

<u>13.3</u>

$$\int \frac{x+1}{x^2-3x+2}\,dx = -\int \frac{2}{x-1}\,dx + \int \frac{3}{x-2}\,dx$$

$$= -2\log(|x-1|) + 3\log(|x-2|) + C_{loc}$$

$$= \log\left(\frac{|x-2|^3}{(x-1)^2}\right) + C_{loc}$$

$$\int \frac{x^2-x+1}{x^3-2x^2+x}\,dx = \int \frac{1}{x}\,dx + \int \frac{1}{(x-1)^2}\,dx$$

$$= \underline{\underline{\log|x| - \frac{1}{x-1} + C_{loc}}}$$

$$\int \frac{x^3}{x+1}\,dx = \int x^2 - x + 1 - \frac{1}{x+1}\,dx$$

$$= \underline{\underline{\frac{1}{3}x^3 - \frac{1}{2}x^2 + x - \log|x+1| + C_{loc}}}$$

$$\int \frac{x^3-2x^2+3}{x^2-3x+2}\,dx = \int x+1 - \frac{2}{x-1} + \frac{3}{x-2}\,dx$$

$$= \underline{\underline{\frac{1}{2}x^2 + x + \log\left(\frac{|x-2|^3}{(x-1)^2}\right) + C_{loc}}}$$

$$\int \frac{x}{(x^2-1)^2}\,dx = \frac{1}{4}\int \frac{1}{(x-1)^2} - \frac{1}{(x+1)^2}\,dx$$

$$= \frac{1}{4}\left(\frac{-1}{x-1} + \frac{1}{x+1}\right)$$

$$= \frac{1}{4}\,\frac{x-1-x-1}{x^2-1} = \underline{\underline{-\frac{1}{2}\,\frac{1}{x^2-1}}} + C_{loc}$$

13.3

$$\int_2^3 \frac{x}{(x^2-1)^2}\,dx = \left[-\frac{1}{2}\,\frac{1}{x^2-1}\right]_{x=2}^{x=3}$$

$$= -\frac{1}{2}\left(\frac{1}{8} - \frac{1}{3}\right)$$

$$= -\frac{1}{2}\left(\frac{3-8}{24}\right) = \underline{\underline{\frac{5}{48}}}$$

$$\int \frac{1}{x^3-x^2+x-1}\,dx \quad :$$

Partialbruchzerlegung:

$$\frac{1}{x^3-x^2+x-1} = \frac{1}{(x-1)(x^2+1)}$$

$$= \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$$

$$= \frac{A(x^2+1) + B(x^2-x) + C(x-1)}{(x-1)(x^2+1)}$$

$$\Rightarrow \begin{array}{l} A + B = 0 \\ -B + C = 0 \\ A - C = 1 \end{array} \quad \Rightarrow \quad 2A = 1 \quad \Rightarrow A = \frac{1}{2}$$

$$\Rightarrow B = -\frac{1}{2} = C$$

$$\Rightarrow \int \frac{1}{x^3-x^2+x-1}\,dx = \frac{1}{2}\int \frac{1}{x-1} - \frac{x+1}{x^2+1}\,dx$$

$$= \frac{1}{2}\left(\log|x-1| - \int \frac{1}{2y}\,dy - \int \frac{1}{x^2+1}\,dx\right)$$

$$= \frac{1}{2}\log|x-1| - \frac{1}{4}\log(x^2+1) - \frac{1}{2}\arctan(x) + C_{loc}$$

13.4

a) $f(x) = e^{x^2} = \sum_{k=0}^{\infty} \frac{1}{k!} (x^2)^k$

$= \sum_{k=0}^{\infty} \frac{1}{k!} x^{2k}$

b) $F(x) = \int_0^x e^{y^2} dy$

$= \int_0^x \sum_{k=0}^{\infty} \frac{1}{k!} y^{2k} dy$

gliedweise Integration

$= \sum_{k=0}^{\infty} \int_0^x \frac{1}{k!} y^{2k} dy$

$= \sum_{k=0}^{\infty} \frac{1}{k!} \frac{1}{2k+1} x^{2k+1}$

c) $G(x) = \int_0^x \cos(y^2) dy$

$= \int_0^x \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} y^{4k} dy$

gliedweise Integration

$= \sum_{k=0}^{\infty} \int_0^x \frac{(-1)^k}{(2k)!} y^{4k} dy$

$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \frac{1}{4k+1} x^{4k+1}$

d) In ähnlicher Weise folgt

$F(x) = \sum_{k=0}^{\infty} \int_0^x \frac{(-1)^k}{(2k+1)!} y^{4k+2} dy$

$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \frac{1}{4k+3} x^{4k+3}$

13.4

e) F ist durch eine Potenzreihe definiert. Solche Funktionen sind immer glatt.

f) Da F durch eine Potenzreihe definiert ist, entspricht die Taylorreihe gerade der Potenzreihe.

g) $T^3_{F, x_0=0}(x) = \frac{1}{0!} \frac{1}{2\cdot 0+1} x^{2\cdot 0+1} + \frac{1}{1!} \cdot \frac{1}{2+1} x^{2+1}$

$= x + \frac{1}{3} x^3$

$\Rightarrow T^3_{F, x_0=0}\left(\frac{1}{10}\right) = \frac{1}{10} + \frac{1}{3} \cdot \frac{1}{10^3}$

$= \frac{300}{3000} + \frac{1}{3000} = \underline{\underline{\frac{301}{3000}}}$

h) $\left| \int_0^{\frac{1}{10}} e^{y^2} dy - T^3_{F, x_0=0}\left(\frac{1}{10}\right) \right|$

$= \left| \frac{F^{(4)}(\xi)}{4!} \cdot \left(\frac{1}{10}\right)^4 \right| = \frac{|F''''(\xi)|}{4!} 10^{-4}$

Wir müssen also zeigen, dass

$|F''''(\xi)| \leq 4! \cdot 10^{-1}$

$F'(x) = e^{x^2}$, $F''(x) = 2x e^{x^2}$

$F'''(x) = 2e^{x^2} + 4x^2 e^{x^2}$

$F''''(x) = 4x e^{x^2} + 8x e^{x^2} + 8x^3 e^{x^2}$

$= (8x^3 + 12x) e^{x^2} = 24\left(\frac{1}{3} x^3 + \frac{1}{2} x\right) e^{x^2}$

$F'''''(x) = (24x^2 + 12) e^{x^2} + (16x^4 + 24x^2) e^{x^2}$

Für $x \in (0, \frac{1}{10})$ ist $F''''' (x) > 0$

$\Rightarrow F''''$ wächst streng monoton auf $(0, \frac{1}{10})$

$\Rightarrow \forall x, y \in (0, \frac{1}{10}) \quad x < y \Rightarrow F''''(x) < F''''(y)$

$\xi < \frac{1}{10} \Rightarrow F''''(\xi) < F''''(\frac{1}{10})$

$$\Rightarrow \frac{|F''''(\xi)|}{4!} \cdot 10^{-4}$$

$$= \frac{F''''(\xi)}{4!} 10^{-4} \qquad (F''''(\xi) \geq 0)$$

$$< \frac{F''''(\frac{1}{10})}{4!} 10^{-4}$$

$$= \frac{\cancel{24} \cdot (\frac{1}{3} \cdot 10^{-3} + \frac{1}{2} 10^{-1}) e^{10^{-2}}}{\cancel{24}} 10^{-4}$$

$$< 10^{-1} \cdot 10^{-4} = 10^{-5} \quad \square$$

13.5

$u : [0,1) \to \mathbb{R}$

$\dot{u}(t) = 2t\, u(t)^2 \qquad \forall t \in [0,1)$

$u(0) = 1$

a) $\Rightarrow \frac{1}{u^2(t)} \dot{u}(t) = 2t \quad \Rightarrow \int \frac{1}{u^2(t)} \dot{u}(t)\, dt = \int 2t\, dt$

$\Rightarrow \int \frac{1}{u^2}\, du = \int 2t\, dt \qquad \begin{pmatrix} u = u(t) \\ du = \dot{u}(t)\, dt \end{pmatrix}$

$\Rightarrow -\frac{1}{u} = t^2 + C_{loc}^{\mathbb{R}}$

$\Rightarrow u(t) = -\frac{1}{t^2 + C_{loc}^{\mathbb{R}}}$

$u(0) = 1 = -\frac{1}{C_{loc}^{\mathbb{R}}} \Rightarrow C_{loc}^{\mathbb{R}} = -1$

$\Rightarrow u(t) = \frac{1}{1-t^2} \quad \square$

b) $\frac{du}{dt} = 2t\, u^2 \quad \Rightarrow \frac{1}{u^2}\, du = 2t\, dt$

$\Rightarrow \int \frac{1}{u^2}\, du = \int 2t\, dt$

Das ist wieder dieselbe Gleichung wie in (a) und führt also zur gleichen Lösung.

c) Wir trennen die Ableitung $\frac{du}{dt}$ in zwei Infinitesimale du und dt auf. Das haben wir aber nicht definiert in der Vorlesung.