Grenzwerte, Lösungen

Bei Fehlern in den Lösungen bitte bei mir melden (nbruhin@ethz.ch).

1. $$\lim_{x\to\infty} \frac{3x^4+2x^2-1}{7x^5+4x} = \lim_{x\to\infty} \frac{\frac{1}{x^5}(3x^4+2x^2-1)}{\frac{1}{x^5}(7x^5+4x)} = \lim_{x\to\infty} \frac{3\frac{1}{x}+2\frac{1}{x^3}-\frac{1}{x^5}}{7+4\frac{1}{x^4}} = \frac{0}{7} = 0$$

2. $$\lim_{x\to 3} e^x \frac{x+7}{x^2+1} = e^3 \frac{3+7}{3^2+1} = e^3 \frac{10}{10} = e^3$$

3. $$\lim_{x\to 0} \frac{\sin(x)}{x} \overset{\text{l'Hopital}}{=} \lim_{x\to 0} \frac{\cos(x)}{1} = 1$$

4. $$\lim_{x\to 0} \frac{\sin(3x)}{\sin(4x)} \overset{\text{l'Hopital}}{=} \lim_{x\to 0} \frac{3\cos(3x)}{4\cos(4x)} = \frac{3}{4}$$

5. $$\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{x^2+x}\right) = \lim_{x\to 0} \frac{(x+1)-1}{x^2+x} = \lim_{x\to 0} \frac{x}{x^2+x} = \lim_{x\to 0} \frac{1}{x+1} = 1$$

6. $$\lim_{x\to 1} \frac{\sqrt{x}-1}{x-1} \overset{\text{l'Hopital}}{=} \lim_{x\to 1} \frac{\frac{1}{2\sqrt{x}}}{1} = \frac{1}{2}$$

7. $$\lim_{x\to\infty} \frac{\sqrt{x}}{e^{\sqrt{x}}} \overset{\text{l'Hopital}}{=} \lim_{x\to\infty} \frac{\frac{1}{2\sqrt{x}}}{\frac{1}{2\sqrt{x}}e^{\sqrt{x}}} = \lim_{x\to\infty} \frac{1}{e^{\sqrt{x}}} = 0$$

8. $$\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$$

   Denn $\sin\left(\frac{1}{x}\right) \in [-1,1]$. Es folgt also

   $$\lim_{x\to 0} x^2 \sin\left(\tfrac{1}{x}\right) \leq \lim_{x\to 0} \quad x^2 = 0$$
   $$\lim_{x\to 0} x^2 \sin\left(\tfrac{1}{x}\right) \geq \lim_{x\to 0} -x^2 = 0$$

   Entsprechend muss der Grenzwert also 0 sein.

9. $$\lim_{x\to 0} \frac{\tan(x)}{x} = \lim_{x\to 0} \frac{\frac{1}{\cos^2(x)}}{1} = 1$$

10. $$\lim_{x\to 0} \sqrt{\frac{1-\cos(x)}{2x}} = \sqrt{\lim_{x\to 0} \frac{1-\cos(x)}{2x}} \overset{\text{l'Hopital}}{=} \sqrt{\lim_{x\to 0} \frac{\sin(x)}{2}} = \sqrt{\frac{0}{2}} = 0$$

11. $$\lim_{x\to\infty} xe^{-\sqrt{x}} \overset{\text{l'Hopital}}{=} \frac{1}{\frac{1}{2\sqrt{x}}e^{\sqrt{x}}} = 2\frac{\sqrt{x}}{e^{\sqrt{x}}} \overset{\text{l'Hopital}}{=} 2\lim_{x\to\infty} \frac{\frac{1}{2\sqrt{x}}}{\frac{1}{2\sqrt{x}}e^{\sqrt{x}}} = 2\lim_{x\to\infty} \frac{1}{e^{\sqrt{x}}} = 0$$

12. $$\lim_{x\to 0} \frac{7^x - \cos(x)}{x} = \lim_{x\to 0} \frac{e^{\log(7)x} - \cos(x)}{x} \overset{\text{l'Hopital}}{=} \lim_{x\to 0} \left(\log(7)e^{\log(7)x} + \sin(x)\right) = \log(7)$$

13. $$\lim_{x\to\infty} x^{\frac{1}{x}} = \lim_{x\to\infty} e^{\log(x)\frac{1}{x}} = \exp\left(\lim_{x\to\infty} \frac{\log(x)}{x}\right) \overset{\text{l'Hopital}}{=} \exp\left(\lim_{x\to\infty} \frac{1}{x}\right) = \exp(0) = 1$$

14. $$\lim_{x\to\infty} \left(\sqrt{x^2+x} - x\right) = \lim_{x\to\infty} \left(\sqrt{x^2+x} - x\right) \frac{\sqrt{x^2+x}+x}{\sqrt{x^2+x}+x} = \lim_{x\to\infty} \frac{(x^2+x)-x^2}{\sqrt{x^2+x}+x}$$
    $$= \lim_{x\to\infty} \frac{x}{x\left(\sqrt{1+\frac{1}{x}}+1\right)} = \lim_{x\to\infty} \frac{1}{\sqrt{1+\frac{1}{x}}+1} = \frac{1}{2}$$

15.
$$\lim_{x\to 0}\left(x^{\sin(x)}\right) = \lim_{x\to 0} e^{\log(x)\sin(x)} = \exp\left(\lim_{x\to 0}(\log(x)\sin(x))\right) = \exp\left(\lim_{x\to 0}\left(\frac{\log(x)}{\frac{1}{\sin(x)}}\right)\right)$$
$$\overset{\text{l'Hopital}}{=} \exp\left(\lim_{x\to 0}\left(\frac{\frac{1}{x}}{-\frac{\cos(x)}{\sin^2(x)}}\right)\right) = \exp\left(-\lim_{x\to 0}\frac{\sin^2(x)}{x\cos(x)}\right)$$
$$= \exp\left(-\lim_{x\to 0}\frac{\sin(x)}{x}\cdot\lim_{x\to 0}\frac{\sin(x)}{\cos(x)}\right) = \exp(-1\cdot 0) = 1$$

16.
$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty}\exp\left(x\log\left(1+\frac{1}{x}\right)\right) = \exp\left(\lim_{x\to\infty}\frac{\log\left(1+\frac{1}{x}\right)}{\frac{1}{x}}\right)$$
$$\overset{\text{l'Hopital}}{=} \exp\left(\lim_{x\to\infty}\left(\frac{\frac{1}{1+\frac{1}{x}}\cdot\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}}\right)\right) = \exp\left(\lim_{x\to\infty}\frac{1}{1+\frac{1}{x}}\right)$$
$$= e$$

17.
$$\lim_{x\to 0^+}\frac{e^{-\frac{1}{x}}}{x} = \lim_{x\to 0^+}\frac{\frac{1}{x}}{e^{\frac{1}{x}}} \overset{\text{l'Hopital}}{=} \lim_{x\to 0^+}\frac{-\frac{1}{x^2}}{-\frac{1}{x^2}e^{\frac{1}{x}}} = \lim_{x\to 0^+}\frac{1}{e^{\frac{1}{x}}} = 0$$